Strassen's support functionals coincide with the quantum functionals

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FOCS 2026, New York, Nov. 11, 2026

Motivation: matrix multiplication exponent

How efficiently can you multiply two $n \times n$ matrices over field $\mathbb{F}$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Straightforward: $\mathcal{O}(n^3)$

[Strassen, 1969]: $\mathcal{O}(n^{2.81})$

Matrix multiplication exponent: $\omega := \inf\{\beta \mid \tilde{\mathcal{O}}(n^\beta) \text{ algorithm exists}\}$

Long history of improvements:
[Schönhage, 1981, Coppersmith and Winograd, 1990, Alman et al., 2025]...

Record(s) before AI: $\omega \leq 2.371...$

Latest: $\omega \leq 9/4 = 2.25$ ([OpenAI, 2026] for $\mathbb{F} = \mathbb{C}$, followed by many expositions for any $\mathbb{F}$)

Conjecture: $\omega = 2$.

Key concept to these results: **tensor rank**

Tensors and their rank

Matrix multiplication is a bilinear map: written as a 3-tensor:

$$\mathrm{MM}_n \coloneqq \sum_{i,j,k=1}^{n} e_{ij} \otimes e_{jk} \otimes e_{ik} \in \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2}.$$

Tensor: multimode array $T = (t_{ijk}) \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$
(Consider $(d = 3)$-tensors for simplicity of presentation)

Tensor rank (decomposition):

- $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \qquad (u_i \in \mathbb{F}^{n_1}, v_i \in \mathbb{F}^{n_2}, w_i \in \mathbb{F}^{n_3})$
- $\mathrm{R}(T) \coloneqq$ minimum r s.t. such decomposition exists

Asymptotic rank:

- $\underset{\sim}{\mathrm{R}}(T) \coloneqq \lim_{m\to\infty} \mathrm{R}(T^{\boxtimes m})^{1/m}$

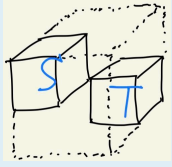
⋆ $\omega = \lim_{n\to\infty} \log_n(\mathrm{R}(\mathrm{MM}_n)) = \log_2 \underset{\sim}{\mathrm{R}}(\mathrm{MM}_2)$ [Strassen, 1980s].

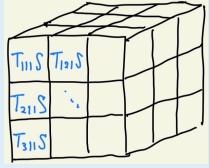
Asymptotic spectrum

Asymptotic spectrum [Strassen, 1980s]

A function $F\colon \{3\text{-tensors}\} \to \mathbb{R}$ is a **spectral point** if

- normalized: $F(\langle 1 \rangle) = 1$
- additive: $F(S \oplus T) = F(S) + F(T)$
- multiplicative: $F(S \boxtimes T) = F(S)F(T)$
- monotone: $F((A \otimes B \otimes C)T) \leq F(T)$





Asymptotic spectrum $\mathcal{X}$: the set of all spectral points

Proposition [Strassen, 1980s]

$$\underset{\sim}{\mathrm{R}}(T) = \max_{F \in \mathcal{X}} F(T)$$

(Moreover, $\mathcal{X}$ completely characterizes "asymptotic restriction".)

Question: Construction of spectral points

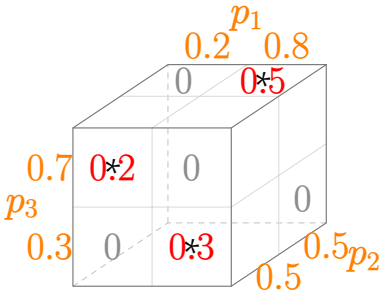
History on spectral point construction

- **[Strassen, 1991]** constructed **support functionals** ζ^θ and *conjectured* that they are spectral points (for every choice of the parameter θ).
  - Straightforward: sub-additive, sub-multiplicative, monotone
- **[Christandl et al., 2018]**, for $\mathbb{F} = \mathbb{C}$, constructed **quantum functionals** F_θ and showed that they are spectral points.
  - These are the only explicitly known spectral points so far
  - Straightforward: super-additive, super-multiplicative, monotone
  - Not easy: sub-additive, sub-multiplicative (proved via representation theory)
  - Also easy: $\zeta^\theta(T) \geq F_\theta(T)$

Our result: For every complex tensor T, it holds that $\zeta^\theta(T) = F_\theta(T)$.

- ⋆ Strassen's 35-year-old conjecture is true for $\mathbb{F} = \mathbb{C}$
- ⋆ New proof that $\zeta^\theta = F_\theta$ is additive and multiplicative, without rep. theory

Support functionals and support polytopes



- Assign a probability p on the support
- Take the marginal distributions (p_1, p_2, p_3)

Support polytope

$\Omega(T) \coloneqq \{(p_1, p_2, p_3) \mid p\text{: prob. on the support}\}$

For a fixed probability distribution $\theta = (\theta_1, \theta_2, \theta_3)$,

Support functional

$$\log \zeta^{\theta}(T) \coloneqq \min_{g_\ell \in \mathrm{GL}_{n_\ell}} \max_{(p_1,p_2,p_3) \in \Omega((g_1 \otimes g_2 \otimes g_3)T)} \sum_{\ell=1}^{3} \theta_\ell H(p_\ell) \quad (H: \text{ Shannon entropy})$$

Note: $\Omega(T)$ is basis-dependent, so optimize over all basis choices.

Entanglement polytopes and q. functionals

Entanglement polytope $\Delta(T)$: "quantum" counterpart of the support polytope

$0 \neq T = (t_{ijk})$: tensor on $\mathbb{F} = \mathbb{C}$.

→ Quantum marginals $(\mu_1(T), \mu_2(T), \mu_3(T))$: p.s.d. Hermitian matrices defined by e.g., $(\mu_1(T))_{ii'} \coloneqq \sum_{jk} t_{ijk}\overline{t_{i'jk}}/\|T\|^2$

Entanglement polytope [Walter et al., 2013]

$$\Delta(T) \coloneqq \overline{\{(\operatorname{spec}(\mu_1(S)), \operatorname{spec}(\mu_2(S)), \operatorname{spec}(\mu_3(S))) \mid S = (g_1 \otimes g_2 \otimes g_3)T, g_\ell \in \mathrm{GL}_{n_\ell}\}}$$

Proposition: $\Delta(T)$ is a (convex) polytope. [Kirwan, 1984, Ness and Mumford, 1984]

Quantum functional

$$\log F_\theta(T) \coloneqq \max_{(p_1,p_2,p_3)\in\Delta(T)} \sum_{\ell=1}^{3} \theta_\ell H(p_\ell)$$

General result: maximin formula

Main theorem

Let $\phi\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \to \overline{\mathbb{R}}$ be

- convex and lower-semicontinuous,
- symmetric: $\phi(P_1 y_1, P_2 y_2, P_3 y_3) = \phi(y_1, y_2, y_3)$ $(P_1, P_2, P_3$: permutations).

Then, for any tensor $0 \neq T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, it holds that

$$\min_{(p_1,p_2,p_3)\in\Delta(T)} \phi(p_1,p_2,p_3) = \max_{g_i\in \mathrm{GL}_{n_i}} \min_{(p_1,p_2,p_3)\in\Omega((g_1\otimes g_2\otimes g_3)T)} \phi(p_1,p_2,p_3).$$

New formula of the form **quantum value** = **optimization of classical value**
(cf. von Neumann vs. Shannon entropy)

Choosing various ϕ produces many known/unknown equations (next page)

Some implications

- $\phi = -\sum_{\ell=1}^{3} \theta_\ell H(p_\ell)$
  Quantum functional $F_\theta(T)$ = **support functionals** $\zeta^\theta(T)$ [New]

- $\phi = \max_\ell -H(p_\ell)$
  Asymptotic slice rank $\underset{\sim}{\mathrm{SR}}(T)$ [Christandl et al., 2018, Christandl et al., 2023]
  = minimum asymptotic vertex cover number of hypergraphs [New]

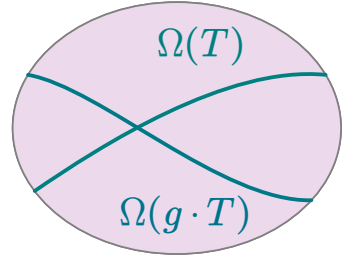
- $\phi = \max_\ell \|p_\ell\|_\infty$
  Two known formulas for the G-**stable rank** [Derksen, 2022]

- $\phi = \|p_1 - \mathbf{1}_n/n\|_1 + \|p_2 - \mathbf{1}_n/n\|_1$
  Non-commutative rank of matrix tuples (e.g., [Fortin and Reutenauer, 2004, Garg et al., 2020, Ivanyos et al., 2017, Hamada and Hirai, 2021, Franks et al., 2023])
  Recent formula in [Hirai, 2026]
  = characterization via maximum vanishing subspaces [Fortin and Reutenauer, 2004]

Proof idea

Recent framework of convex optimization over manifolds [Hirai, 2025]
cf. Non-commutative optimization [Bürgisser et al., 2019]

- Optimization over $\Delta(T)$
  = minimization over the entire manifold
- Optimization over $\Omega(T)$
  = minimization over its submanifold



Key: a **duality theorem** [Hirai, 2025] that rewrites the minimization problem as another maximization problem

Proof structure (not strictly correct):

$$\min_{x \in \text{manifold}} \overset{\text{duality}}{=} \max_{y \in \text{manifold}} \overset{\text{split max.}}{=} \max_{\text{for all submanifold}} \max_{y \in \text{submanifold}} \overset{\text{duality}}{=} \max_{\text{for all submanifold}} \min_{x \in \text{submanifold}}$$

Summary

Our result: quantum functionals F_θ = support functionals ζ^θ (for $\mathbb{F} = \mathbb{C}$)

This is given by a new maximin formula on convex optimization over the polytopes:

$$\min_{(p_1,p_2,p_3)\in\Delta(T)} \phi(p_1,p_2,p_3) = \max_{g_i\in\mathrm{GL}_{n_i}} \min_{(p_1,p_2,p_3)\in\Omega((g_1\otimes g_2\otimes g_3)T)} \phi(p_1,p_2,p_3),$$

which also captures

- asymptotic slice rank,
- G-stable rank,
- non-commutative rank.

Technical tool: Recent framework on convex optimization over manifolds.

Thank you!

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