Entanglement Polytopes, Algorithms, and Applications

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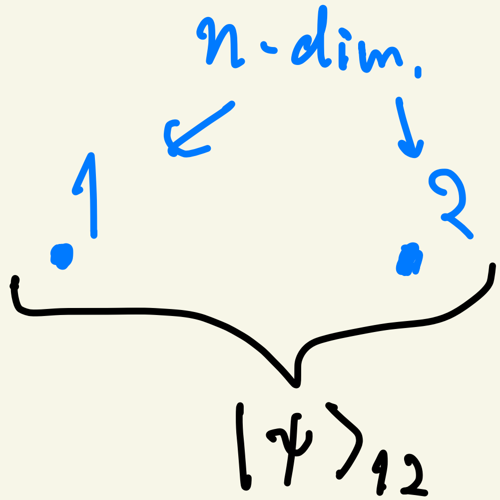
Entanglement between two particles

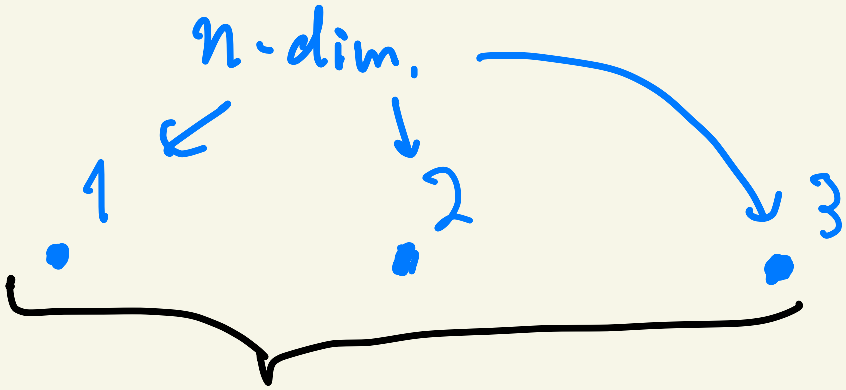


- Pure state $|\psi\rangle_{12} = \sum_{ij} \psi_{ij} |ij\rangle_{12} \Leftrightarrow$ Matrix (ψ_{ij})
- Single-partite states $\rho_1(\psi) = \mathrm{tr}_2\left[|\psi\rangle\langle\psi|\right]$, $\rho_2(\psi) = \dots$
- Entanglement of $|\psi\rangle$ is characterized by

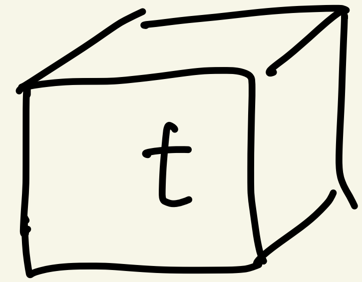
$$\lambda(\rho_1(\psi)) = \lambda(\rho_2(\psi)) \quad \begin{pmatrix} = (1,0,\dots,0) \Leftrightarrow \text{separable} \end{pmatrix}$$

$$\left(= \frac{\mathbb{1}}{n} \quad \Leftrightarrow \text{max. entangled} \right)$$

Entanglement between three particles

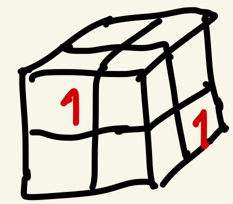


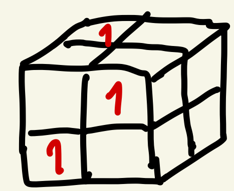
$|t\rangle_{123}$

(3D array) 

- $|t\rangle_{123} = \sum_{ijk} t_{ijk} |ijk\rangle_{123} \Leftrightarrow$ Tensor $t = (t_{ijk})$

- Different types of entanglement

Ex. $n=2$: $|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \Leftrightarrow \frac{1}{\sqrt{2}}$ 

$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \Leftrightarrow \frac{1}{\sqrt{3}}$ 

How can we show that they are essentially different?

Entanglement polytope [Walter et al. '14]

invertible $(A, B, C \in GL_n)$

SLOCC: • $|s\rangle = (A \otimes B \otimes C)|t\rangle$ (with normalization)

• $|s\rangle \sim_{SLOCC} |t\rangle \Leftrightarrow \exists A, B, C$ s.t. above holds.

<u>Entanglement polytope</u>

$$\Delta(t) := \left\{ (\lambda(\rho_1(s)), \lambda(\rho_2(s)), \lambda(\rho_3(s))) \;\middle|\; |s\rangle \in \overline{(GL_n \otimes GL_n \otimes GL_n)|t\rangle} \right\}$$

Fact: $\Delta(t)$ is a polytope! [Ness-Mumford '84, Kirwan '84, ...]

- $|s\rangle \sim_{SLOCC} |t\rangle \Rightarrow \Delta(s) = \Delta(t)$

  $\rightsquigarrow |GHZ\rangle$ and $|W\rangle$ are not SLOCC-equivalent.

- One-body quantum marginal problem: $(p_1, p_2, p_3) \in \Delta(t)$? generic

Quantum functional [Christandl, Vrana, Zuiddam '18]

Fix $\theta = (\theta_1, \theta_2, \theta_3)$ $\left(\theta_i \geq 0,\ \sum_i \theta_i = 1\right)$ [typically, $\theta = (1/3, 1/3, 1/3)$]

- $$E_\theta(t) := \max_{(p_1,p_2,p_3) \in \Delta(t)} \sum_{i=1}^{3} \theta_i H(p_i)$$

  H: Shannon entropy

  $$= \sup_{|s\rangle \sim \mathrm{sloc}\,|t\rangle} \Phi_\theta(s) \quad \left(\Phi_\theta(s) := \sum_i \theta_i H(\rho_i(s))\right)$$

  H: von Neumann entropy

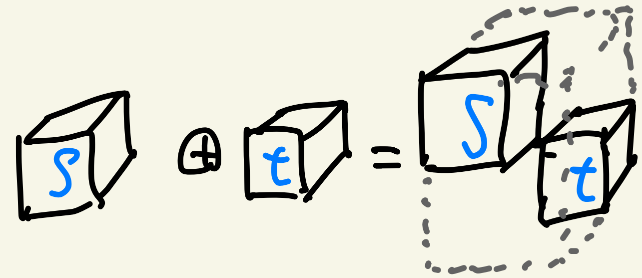
- $F_\theta(t) := 2^{E_\theta(t)}$ ← quantum functional

> Recall: $\Delta(t) := \left\{ \left(\lambda(\rho_A(s)), \lambda(\rho_B(s)), \lambda(\rho_C(s))\right) \,\middle|\, |s\rangle \in \overline{(GL_n \otimes GL_n \otimes GL_n)|t\rangle} \right\}$

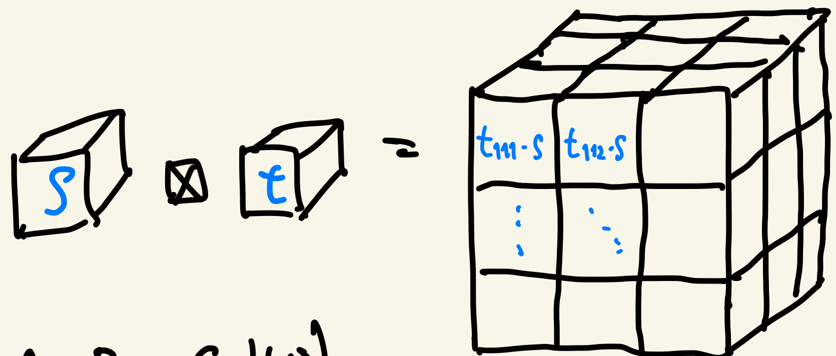
Quantum functional: applications

Properties [Christandl, Vrana, Zuiddam '18]

- $F_\theta\left(\boxed{1}\right) = 1$
- $F_\theta(s \oplus t) = F_\theta(s) + F_\theta(t)$



- $F_\theta(s \boxtimes t) = F_\theta(s)\, F_\theta(t)$



- $s \leq t \Rightarrow F_\theta(s) \leq F_\theta(t)$

$\left(s \leq t \Leftrightarrow \exists A, B, C,\ |s\rangle = (A \otimes B \otimes C)|t\rangle\right)$

not necessarily invertible

Works as an obstruction for asymptotic restriction

$s^{\boxtimes n} \leq t^{\boxtimes \alpha n} \Rightarrow \alpha \geq \log F_\theta(s) / \log F_\theta(t) = E_\theta(s)/E_\theta(t)$

- Provides lower(upper) bounds on asymptotic (sub)rank of tensors
- How many copies of $|t\rangle$ is (at least) needed to create $|s\rangle$?

Quantum functional computation

$$\boxed{\text{Want to compute: } E_\theta(t) = \max_{(p_1, p_2, p_3) \in \Delta(t)} \sum_i \theta_i H(p_i) = \sup_{|s\rangle \sim \mathrm{sLOCC}(t)} \Phi_\theta(s)}$$

Concave maximization over a polytope ... easy???

No! $\Delta(t)$ is very complex!

Computation of $\Delta(t)$ [van den Berg et al., '25] is only practical up to $n = 4$.

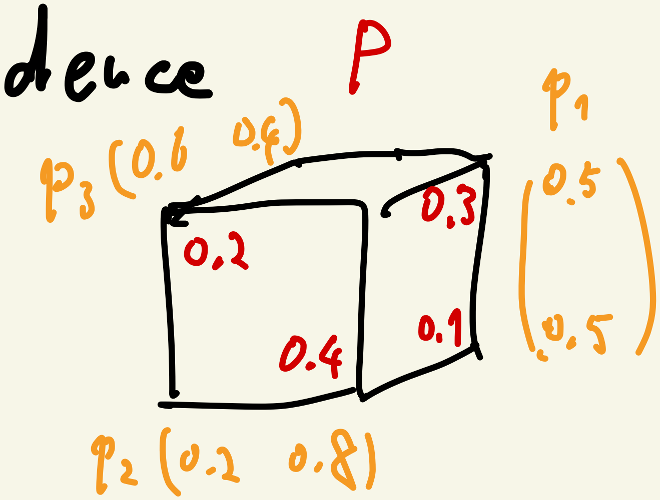
Our algorithm: Entropic Tensor Scaling

$$|t_{k+1}\rangle := \left(\rho_1(t_k)^{-\frac{\theta_1}{2}} \otimes \rho_2(t_k)^{-\frac{\theta_2}{2}} \otimes \rho_3(t_k)^{-\frac{\theta_3}{2}}\right) |t_k\rangle$$

Thm. [Doğan-S-Walter] $E_\theta(t_0) - \Phi_\theta(t_k) = O(k^{-1})$.

First efficient algorithm for quantum functional computation!

Classical-quantum correspondence



Support polytope

$$\Omega(t) :=$$

$$\{(p_1, p_2, p_3) : \text{marginal of prob. dist. } P \mid P_{ijk} > 0 \Rightarrow t_{ijk} \neq 0\}$$

"classical" version of entanglement polytope.

Support functional [Strassen '80s]

$$\zeta^\theta(t) := \min_{|s\rangle \sim \mathrm{SLOCC}|t\rangle} \ \max_{(p_1, p_2, p_3) \in \Omega(s)} \sum_i \theta_i H(p_i)$$

$\boxed{\text{Thm. [Doğan-S-Walter] } F_\theta(t) = \zeta^\theta(t)}$

Generalizes to any symmetric concave function (in place of entropy).

THANK YOU!